SAMPLE CONTENT

Basics of CALCULUS Simplified

A valuable resource for students of Std. XI and XII

$$P'(t) = \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t}$$

Basic concepts of Calculus can be understood by analyzing any moving object like bike, car, train, aeroplane and so on.

Target Publications Pvt. Ltd.

Basics of

CALCULUS

Simplified

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Printed at: **Print to Print,** Mumbai

P.O. No. 97060

10410_11030_JUP

Some guidelines for reading this book:

- All the chapters in this book are arranged in a logical order, so that readers can read them sequentially.
- The figures and graphs are placed on the same page or the facing page where their description appears. This eliminates the need for numbering the figures and connecting them to the texts.
- Readers may find repetition of some 'text material' in different chapters. This was seen as a better option than to ask readers to go back and forth whenever a reference is made to a text from a different chapter. Repetition has also been used to highlight and drive home the importance of a particular 'text'.
- Study of calculus should not be about memorizing differentiation and integration formulae. The main focus of the readers should be on understanding and appreciating the fundamental logic and basic concepts of calculus.
- This is not intended to be a textbook of calculus. The main objective of this book is to clearly explain basic concepts of calculus. Accordingly more emphasis was given to include different interpretations of basic ideas of calculus, rather than incorporating proofs of all the theorems of calculus. Hence the readers may find that some related sub-topics of few subjects are not included in this book or some topics have not been covered in as much detail.
- **Conceptual understanding** and **analytical abilities** are the two most important qualities essential to attain a certain level of proficiency in any subject. This is however particularly important in case of a subject like calculus. **Conceptual understanding** of calculus basics can be developed by reading theory books like the one you are about to read. However, **Analytical abilities** cannot be developed by merely reading books. **The only way to develop analytical skills in calculus is to practice solving as many different calculus problems as possible from text books or from any other source.**
- Readers are welcome to send their suggestions, comments and feedback about this book to following email address.
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PREFACE

Students passing out of school and entering Junior College approach the subject of Calculus with a great deal of nervousness, if not an outright sense of dread. The subject is looked upon more as a **challenge** to somehow overcome in college than gaining knowledge of what may be described as the **crown jewel of mathematics**. Advanced text books on the subject tend to get overly technical, reinforcing the adverse impression among the beginners that Calculus is a **tough subject**, and the smart way is to somehow 'clear' the subject by going over it mechanically with the help of guides.

I have carried within me this disturbing feeling for long, that instead of being seen as a fascinating subject that should fire students' imagination, Calculus is just seen as a troublesome subject.

This book on the basics of Calculus, if this presentation may be so described, is thus a humble attempt to introduce Calculus to youngsters just out of school, in a language with which they would feel at ease, helping them with liberal use of examples drawn from our day to day life. All the topics are explained in a structured manner so that students do not miss out on the fundamental logic and basic concepts of Calculus. The physical, graphical, algebraic and geometric interpretations of basic ideas of Calculus are explained in such great detail that students can look at Calculus from a refreshing viewpoint and dispel the impression that it is a tough subject.

The basic approach followed here for getting the subject across was to let it unfold itself as a flowing narrative. Yet care has been taken to ensure that no part of the contents would suffer from errors in the exposition of the basics.

From 'thinking' to the 'finished draft' of the book was a long and difficult journey, which was only made possible with the help from many friends. I am extremely happy to express my gratitude to all of them.

Prof. Dr. Sharad S. Sane from IIT-Bombay checked the initial draft of the first few chapters and gave his word of encouragement to undertake this project. He also checked the final draft and gave several important suggestions. **Prof. Dr. Vasant M. Kane**, former H.O.D. Mathematics and Vice Principal of RKT College, **Prof. Prakash G. Dixit**, H.O.D. Statistics, Modern College, **Prof. A.V. Rayarikar**, former H.O.D. Mathematics, Modern College, **Prof. Mrs. S. A. Joshi** from Ruia College and **Mr. A.B. More** extended all the help in editing the book and gave extremely useful and valuable comments for improving the text.

Mr. A.L. Narasimhan, a colleague and a friend for several years, extended his help in editing several drafts at various stages. His facility with language has greatly contributed to the flowing text that the readers may notice.

My acknowledgement of gratitude may not be complete without mentioning my friends **Mr. J. V. Kadhe** and **Mr. V. A. Patwardhan**, who spared no efforts to lead me to valuable contacts, which were of great help in editing the book.

Lastly I would like to submit to such of my readers who are learned scholars of mathematics that I am acutely aware that this book, both in its contents and style, will seem to them to be rather elementary and incomplete, and will lend itself to much improvement. I seek their indulgence and wish to stress again that, this book is essentially addressed to beginners in their study of Calculus, and if it succeeds in interesting them and making them seek closer study through more comprehensive books, I shall consider my labour amply rewarded.

A. A. Tilak

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02 Derivative (Physical interpretation)

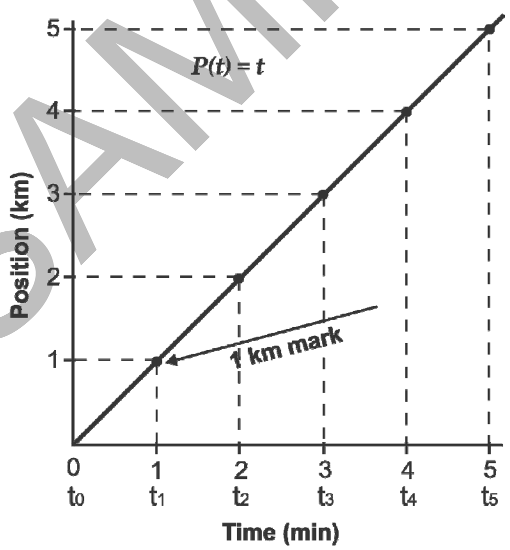
Introduction

Let us take a look at the physical interpretation of a derivative. **Derivative is a measure of 'rate of change'**. Change occurs in many things around us like weather, positions of sun and moon, day to day temperatures, share index, exchange rate of Rupee and Dollar and so on. The most fundamental change is **'motion'**, i.e. the **change in position of an object with respect to time.** Let us analyze this **'change'** by taking a few simple examples.

2.1 Concept of Instantaneous Velocity

Suppose we are driving on a straight road in a car, whose **position** at any time is **proportional to the time elapsed** from the point of starting. Hence, the **position function** for such a car would be $\boldsymbol{P(t) = t}$.

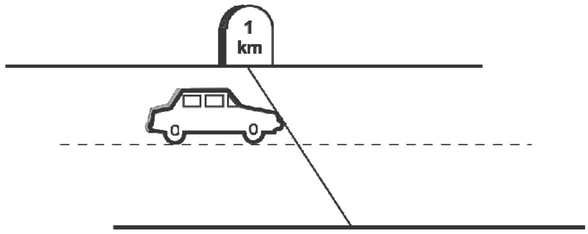
We can see from the graph that the **position** of a car which is shown on **Y axis** is the **'output variable'**, and the **time** on **X axis** is the **'input variable'** of this function. The output variable or the position of a car changes as the input variable or the time changes.



In this case, starting from the **0 km** mark at time t_0 **= 0 min,** the car will be at **1 km** mark at time t_1 **= 1 min** and at **2 km** mark at t_2 **= 2 min** and at **3 km** mark at t_3 **= 3 min** and so on, as can be seen from the graph.

We can observe that, the car is moving with a certain speed all the time, from the fact that starting from the **zero km** position; it reaches the **1 km** mark in **1 min**, **2 km** mark in **2 min, 3 km** mark in **3 min,** and so on. Hence it is quite clear that the car is moving at a certain speed all the time.

Our objective is to find the speed or **velocity of the car** exactly at the **instant of time** t_1 **= 1 min,** when the car is at **1 km** mark. Let us find out some more information about the status of the car when it is at **1 km** mark, before proceeding further to calculate its speed or velocity. Let us mount a camera in front of the **1 km** mark and set it so as to take a **photograph** at the instant of time t_1 **= 1 min** when the car reaches **1 km** mark. **To our surprise, this moving car appears stationary in that photograph as seen below !!**



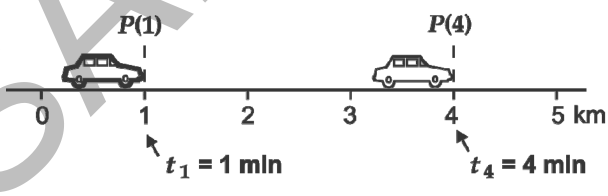
Hence, we have a complex problem here to find the speed or velocity of the car at the instant **of time** t_1 **= 1 min** when it is at the **1 km** mark, where it appears to be **stationary**. We are all familiar with finding speed or velocity of moving objects. The standard method for finding average speed or **velocity** is to **divide the change in position by change in time**. However, for the car that appears **stationary** at the instant of time t_1 **= 1 min,** when it is at the **1 km** mark, there is **neither change in position nor change in time**.

Hence, finding **speed** or **velocity** of the car at the **instant of time t_1 = 1 min** when it is at **1 km** mark is not quite so straight forward.

Let us follow a simple idea to get the answer. Starting from an instant of time of **t_1 = 1 min**, let us select an arbitrary **time interval** and find **distance travelled** by the car during that **time interval** by using its position function **$P(t) = t$**. Let us find out **average velocity** of the car for that **time interval** by dividing **distance travelled** by the **time interval**. Then let us keep on reducing that **time interval** progressively and find out **average velocities** for **shorter and shorter time intervals**.

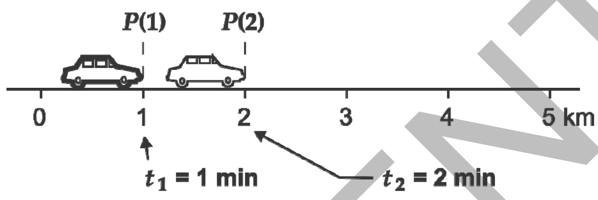
Finally we can make the **time interval extremely short** (say equal to **0.000,000,001 min** or **1/16th** of **1 micro second**) and calculate the **average velocity** for this **fraction of micro second** time interval. This **fraction of micro second** time interval is so close to the instant of time **t_1 = 1 min**, that the average velocity calculated for this **fraction of micro second** time interval can be considered virtually equal to the **velocity** of the car at the '**instant of time t_1 = 1 min'**, when it is at the **1 km** mark. And we can call this the **'velocity at the instant'** or the **'Instantaneous Velocity'** of the car at the **'instant of time t_1 = 1 min'**, when it is at the **1 km** mark.

We can start with arbitrary time interval between **t_1 = 1 min** and **t_4 = 4 min** to calculate average velocity of the car for a time interval of **3 min**. The car travels from **1 km** mark to **4 km** mark i.e. from position $P(1)$ to position $P(4)$ in $(t_4 - t_1)$ time.



Avg. vel. = $\frac{P(4)-P(1)}{t_4-t_1} = \frac{4-1}{4-1}$ **= 1 km/min.**

Let us find avg. velocity of the car for an even shorter time interval. Let us consider a time interval between **t_1 = 1 min** and **t_2 = 2 min** to calculate average velocity of the car for a time interval of **1 min.** The car travels from **1 km** mark to the **2 km** mark i.e. from position $P(1)$ to position $P(2)$ in $(t_2 - t_1)$ time.



Avg. vel. = $\frac{P(2)-P(1)}{t_2-t_1} = \frac{2-1}{2-1}$ **= 1 km/min.**

Now let us make the time intervals progressively shorter and find average velocities of the car for these different time intervals by following a similar procedure for the following three cases.

Avg. vel. = $\frac{P(1.1)-P(1)}{t_{1.1}-t_1} = \frac{1.1-1}{1.1-1}$ **= 1 km/min.**

Avg. vel. = $\frac{P(1.01)-P(1)}{t_{1.01}-t_1} = \frac{1.01-1}{1.01-1}$ **= 1 km/min.**

Avg. vel.= $\frac{P(1.001)-P(1)}{t_{1.001}-t_1} = \frac{1.001-1}{1.001-1}$ **= 1 km/min.**

Let us tabulate all the values of average velocities of the car for different time intervals for our analysis.

Start position time= t_1 (min)	End Position time= t_x (min)	Time Interval $(t_x - t_1)$ (min)	Average velocity (km/min)
1	4	3	1
1	2	1	1
1	1.1	0.1	1
1	1.01	0.01	1
1	1.001	0.001	1

This table leads us to conclude that the average velocities of the car for different time intervals starting from instant of time **t_1 = 1 min** remain **constant** at **1 km/min,** even after the time intervals are made progressively shorter.

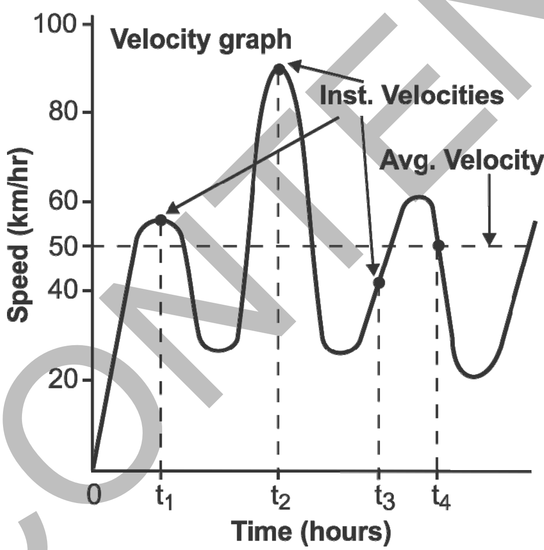
Now if we further reduce this time interval to say **0.000,000,001 min** or **1/16th** of **1 micro second,** then the distance travelled by the car during this short time interval would be equal to **0.000,000,001 km**. Hence, average velocity of the car will still remain same as **1 km/min**. This **fraction of micro second** time interval that is used to calculate average velocity, is so close to the instant of time t_1 **= 1 min**, that the average velocity calculated for this **fraction of micro second** time interval can be considered virtually equal to the **speed** or **velocity** of the car at the '**instant of time** t_1 **= 1 min'**, when it is at the **1 km** mark. Hence we can confidently conclude that, this average velocity value of **1 km/min** is in fact the **instantaneous velocity** of the car at the '**instant of time** t_1 **= 1 min'**, when the car is at the **1 km** mark.

Now we can conclusively state that at the **instant of time** t_1 **= 1 min**, when the car is at the **1 km** mark, it has an **instantaneous velocity** of **1 km/min**. **The car may appear to be stationary at one place at that instant of time** t_1 **= 1 min, when it is at the 1 km mark (as seen in its photograph), but the idea that the car can have a speed at that instant, even when it appears stationary, leads us to the concept of 'Instantaneous speed' or 'Instantaneous velocity'.**

We can also observe that, when you reduce the **time interval** to an **infinitely small value**, then the value of **average velocity for that time interval virtually becomes equal to the value of instantaneous velocity.** (However, in this example, since the car is moving at a constant velocity of **1 km/min**, the values of average velocity for a longer time interval and instantaneous velocity for an extremely short time interval are the same.)

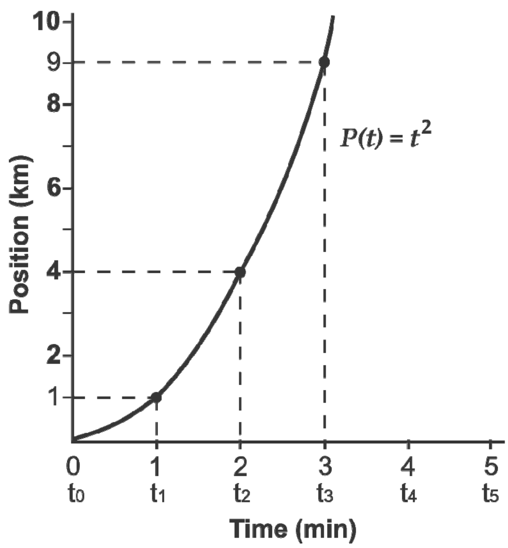
Let us look at another example to further clarify the **difference** between **Average velocity** and **Instantaneous velocity**.

When you travel from Pune to Mumbai, a distance of **200 km.** in **4 hours**, then your **average velocity or average speed** over that distance can be calculated as equal to **50 km/hr**. But on your way to Mumbai, when you observe the speedometer of your car, you will see it moving all the time between as low as **20 km/hr** to as high as **90 km/hr** and it shows various values in between at different instants of time. These are the values of **instantaneous velocities** of the car at those different instants of time.



2.2 Concept of Derivative

Let us consider another example where we are driving on a straight road in a **different car**, whose **position** at any time is given by the **square of the time elapsed**. Hence the **position function** for such a car would be $P(t) = t^2$. **The position graph** for this function is given below.



As we have stated earlier, **position function 'P'** for the car is related to square of time or equal to 't^2', i.e. $P(t) = t^2$

We can see that the **position** of a car which is shown on **Y axis** is the **'output variable'**, and the **time** on **X axis** is the **'input variable'** of this function. The output variable or the position of a car changes as the input variable or the time changes.

Hence, starting at t_0 **= 0 min**, the car starts moving from **0 km** mark.

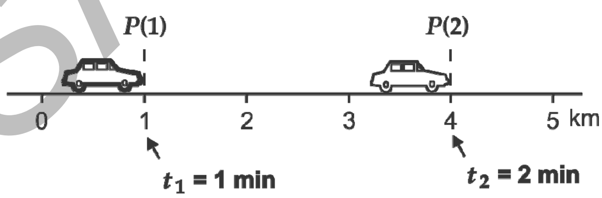
And at time t_1 **= 1 min**, the position of the car will be at **1 km** mark.

And at time t_2 **= 2 min**, the position of the car will be at **4 km** mark.

And at time t_3 **= 3 min**, the position of the car will be at **9 km** mark and so on, as shown in the graph.

Once again, our objective is to find the **instantaneous velocity** of this **different car** at the **instant of time** t_1 **= 1 min** when it is at the **1 km** mark by using the position function $P(t) = t^2$. Let us use a similar logic of finding average velocities for progressively shorter **time intervals**, from the instant of time of t_1 **= 1 min**, and finally reach a **'fraction of a micro second'** time interval so that average velocity calculated for that **'fraction of a micro second'** time interval will be virtually equal to the value of instantaneous velocity at the instant of time t_1 **= 1 min** when the car is at the **1 km** mark.

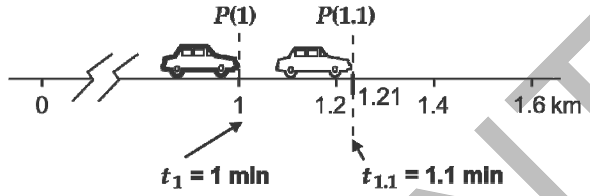
Let us first calculate the **average velocity** of the car for an arbitrary time interval, namely between t_1 **= 1** and t_2 **= 2 min**.



Avg. Vel. = $\dfrac{P(2)-P(1)}{t_2-t_1} = \dfrac{2^2-1^2}{2-1} = \dfrac{4-1}{1}$

= 3 km/min.

Let us find the **average velocity** of the car for a shorter time interval, namely between t_1 **= 1** and $t_{1.1}$**= 1.1 min.**



Avg. Vel. = $\dfrac{P(1.1)-P(1)}{t_{1.1}-t_1} = \dfrac{1.1^2-1^2}{1.1-1} = \dfrac{1.21-1}{0.1}$

= 2.1 km/min.

Let us now find **average velocity** of the car for an even shorter time interval, namely between t_1 **= 1** and $t_{1.01}$**= 1.01 min.**

Avg. Vel. = $\dfrac{P(1.01)-P(1)}{t_{1.01}-t_1} = \dfrac{1.01^2-1^2}{1.01-1}$

$= \dfrac{1.0201-1}{0.01}$

= 2.01 km/min.

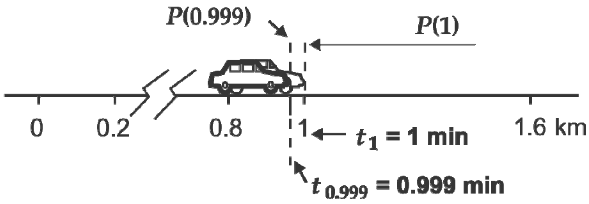
Let us go further and find the average velocity of the car for an even shorter time interval, namely between t_1 **= 1** and $t_{1.001}$**= 1.001 min**.

Avg. Vel. =

$= \dfrac{P(1.001)-P(1)}{t_{1.001}-t_1} = \dfrac{1.001^2-1^2}{1.001-1} = \dfrac{1.002001-1}{0.001}$

= 2.001 km/min.

Now let us find average velocity of the car between $t_{0.999}$**= 0.999** and t_1 **= 1 min**. This is again a time interval of **0.001 min,** where the end position is at time t_1 **= 1 min.** (In all of the earlier examples, t_1 **= 1 min** or **1 km** was the start position, where we calculated average velocities of the car by taking progressively shorter time intervals on the positive side of time t_1 **= 1 min** or **1 km** position. But now let us calculate average velocity of the car by taking a shorter time interval on the negative side of time t_1 **= 1 min** to check the average velocity on the negative side of **1 km mark).**



Avg. Vel. =

$$= \frac{P(1) - P(0.999)}{t_1 - t_{0.999}} = \frac{1^2 - 0.999^2}{1 - 0.999} = \frac{1 - 0.998001}{0.001}$$

= 1.999 km/min.

Let us tabulate all the values of average velocities calculated for progressively shorter time intervals for the analysis.

Start position time = t_1 (min)	End position time = t_x (min)	Time interval ($t_x - t_1$) (min)	Average velocity (km/min)
1	2	1	3
1	1.1	0.1	2.1
1	1.01	0.01	2.01
1	1.001	0.001	2.001
0.999	1	0.001	1.999

We can see that, as we take smaller time intervals, we get **different values of average velocities** of the car. And as we make these time intervals shorter and shorter, we can see a **trend** in these average velocity values to progressively close on to a figure of **2 km/min** from both positive and negative sides.

Now we can visualize that the avg. velocity value will become almost equal to **2 km/min**, when the **time interval** is further reduced to smaller than a **fraction of a micro second** level. **(**Let us call it as **time interval *Δt***, which is **'very very very' small)**. This smaller than a **fraction of a micro second** time interval ***Δt*** is so close to the **instant of time t_1 = 1 min** that, the average velocity value of **2 km/min** can be virtually considered as the value of **instantaneous velocity** of the car at the **instant of time t_1 = 1 min** when it is at **1 km mark**.

[We can now understand the concept of a 'very very' small time interval, which is normally denoted by a symbol as *Δt* or *dt*. (Both these symbols are pronounced as '*delta t*').

For our calculations we will consider *'Δt→0'* or *'dt→0'*.The 'arrow' between '*Δt or dt*' and '*0*' means '*Δt or dt*' is approaching a value which is so extremely small that it is almost equal to zero (but never equal to zero).]

Similarly we can calculate several values of **instantaneous velocities** of the car at different **instants of time**, namely at ***t* = 0, *t* = 0.6, *t* = 1, *t* = 1.4, *t* = 2** and ***t* = 3 min** and tabulate these values for our analysis.

Instant of time (min) t	Instantaneous velocity (km/min) V
0	0
0.6	1.2
1	2
1.4	2.8
2	4
3	6

[*Note:- A sample of instantaneous velocity calculation details for the car when it is at instants of time t = 1.4, t = 2 and t = 3 min is given in para 2.6 at the end of this chapter for your information.*]

We can observe that the values of **instantaneous velocities** of the car are **different** at different **instants of time**.

We can also see a **clear pattern** between the **values of instantaneous velocities** and different **instants of time.** And the pattern that emerges here is that, the **instantaneous velocity** is exactly **twice the instant of time** at which it is measured.

Hence, if the car is moving with a velocity so that its **position** on the road at **any instant of time** is denoted by **square of the time** (i.e. t^2), then its **instantaneous velocity** at any instant of time can be calculated as **twice the value of time** (i.e. $2t$).

If the position function for a car is stated as $P(t)$ = t^2, then we can deduce that, a function for its inst. velocity is $V(t)$ = $2t$.

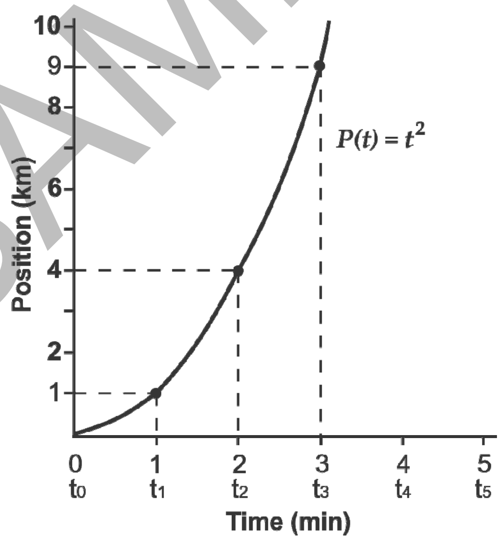
Velocity function $V(t)$ = $2t$ is called the 'Derivative function' or simply the 'derivative' of the position function $P(t)$ = t^2. And the **process of finding the derivative** is called **'Differentiation'**.

Once we have this formula, we need not go through a laborious process to find the average velocities for shorter and shorter time intervals, to finally arrive at the correct value of instantaneous velocity. Instead, we can directly calculate the value of instantaneous velocity by this formula at any instant of time.

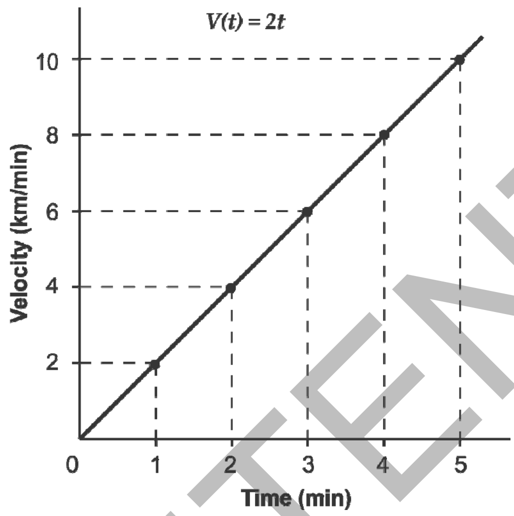
When you know the position function of a car as $P(t)$ = t^2, then deriving its instantaneous velocity at any instant of time t, by a simple formula like inst. velocity $V(t)$ = $2t$ is the power of the 'Differentiation' process.

We can now repeat the position graph for the function $P(t)$ = t^2, and also draw the velocity (Derivative) function graph using different values of instantaneous velocities calculated earlier for comparative study of these two graphs.

Position function Graph:-

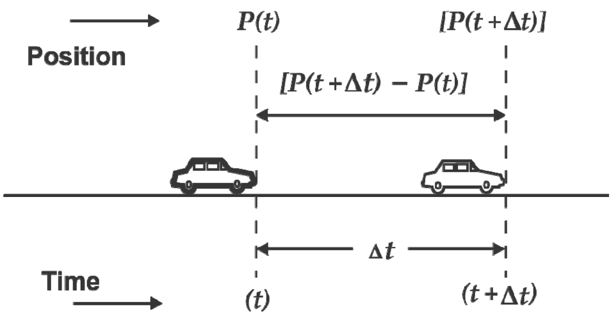


Velocity (Derivative) function Graph:-



2.3 Meaning of a Derivative

Let us review the basic **differentiation process** which was evolved by computing average velocities for shorter and shorter time intervals and ultimately finding the **derivative** or the **instantaneous velocity** for a car at a particular **instant of time t**, for an infinitely short time interval of time Δt. The recap of this process is given below in greater detail.



When we want to compute instantaneous velocity of the car at the instant of time t, we should first compute the **position** of the car at time $(t+\Delta t)$, where Δt is a very small increment of time. (That **position** is $P(t+\Delta t)$). And then we should subtract the **position** where the car was at time(t), (That **position** is $P(t)$), to find the **net distance** travelled by the car in the short interval of time Δt. (That **net distance** is $[P(t+\Delta t)-P(t)]$). Finally, we should divide this net distance travelled, by the short time interval Δt, to get the average velocity for a time interval between time (t) and $(t + \Delta t)$.

And we repeat this process of computing average velocities for smaller and smaller values of time Δt. And as the value of Δt approaches very close to zero (which is symbolically stated as $\Delta t \to 0$), that will give us the **derivative** or the **instantaneous velocity** of a car at the instant of time t. **This is the limiting process of a Differentiation.**

This whole process of finding a **derivative** or an **instantaneous velocity** can be represented by the following general expression.

Instantaneous Velocity or Derivative of a position function $P(t)$ at time t =

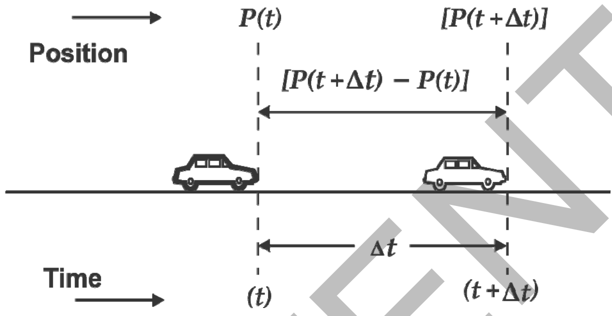
$$= \frac{P(t+\Delta t) - P(t)}{\Delta t} \quad \text{as } \Delta t \to 0$$

(We can observe here that, the basic process of finding a derivative is actually a process of subtraction and division.)

What we have really done here to find the **'derivative'** or **'instantaneous velocity'** of a moving car is to find **a corresponding change in the position of a car** and divide it by an **extremely small change in time**. As we know, when we divide the **'change in position'** of a car by **'change in time'**, then we will get the **'rate of change'** in the position of the car.

Hence, the **derivative** or **instantaneous velocity** of the car can be defined as the **'rate of change'** in its **position**.

(As we have seen earlier, the **'rate of change'** in position of a car means the same as the **'velocity'** of a car. That is why we said that, when you take a **derivative** of the **position function**, then you get the **velocity function.)**
We have used the example of **instantaneous velocity** of a car as the **physical interpretation** of a **derivative** to explain and understand the **concept of a derivative.** Hence we defined **derivative** as a **'rate of change'** in the **position** of a car, which actually is the **definition of the instantaneous velocity**. However, having understood the concept of a derivative, we can now take the **first step** towards **generalizing the definition** of a derivative or instantaneous velocity as follows.



We have stated earlier that the position function $P(t)$ = t^2 is a function where its **'output variable' $P(t)$** represents the **position of a car**. However we can also visualize this **'output variable'** to represent the **'position of any moving object'** rather than restricting it to the example of the **position of a car**. Hence **'Derivative'** or **'Instantaneous velocity'** can also be defined as the **'rate of change'** in the **'position of any moving object'**.

In the example seen earlier, we have considered the position function $P(t)$ = t^2 as a function where the **position of a car $P(t)$** is an **'output variable'**, while the **time t** is an **'input variable'** to construct the definition of a derivative. However we can now consider any general function $y = f(x)$, where value of its **'output variable y'** depends on the varying value of the **'input variable x '** to construct the **general definition** of a **derivative**.

Hence, a **derivative** can be defined as the **'rate of change'** in an **'output variable y'** of any general function $y = f(x)$, representing any of the different **'output variables'** like **'temperature'** or **'area of a circle'** or **'sales volume'** or **'profit'** or **'population'** or any other dependant output variable, rather than restricting it only to the **'position of a car'** or to the **'position of any moving object'**.

All of our earlier examples have used **'rate of change'** in **'output variable y'** with respect to **'input variable x'** of **time**. However we can also analyze **'rate of change'** in '**output**

variable y' with respect to any other **'input variable x**' other than **time**. Hence an **'input variable x**' of the function can also represent **'days'** or **'radius of a circle'** or **'product price'** or **'number of units'** or **'year'** or any other independent input variable, rather than restricting it to the **'time in minutes'**.

We are now ready to take the **final step** to state the **general definition of a derivative**. Derivative is a **'rate of change'** in an **'output variable'** of any function.

(In short, Derivative is ROC).

2.4 Fundamental definition of a derivative

Let us now use the earlier expression of instantaneous velocity or derivative of a position function to arrive at the fundamental definition of a derivative.

Instantaneous Velocity or Derivative of a position function $P(t)$ at time t =

$$= \frac{P(t+\Delta t)-P(t)}{\Delta t} \quad \text{as } \Delta t \to 0$$

In this expression, the denominator Δt is an extremely small change in time. The time t is the **input variable.** Hence we can also redefine the denominator as an **'Extremely small change in the input variable'**.

In this same expression, the numerator $[P(t+\Delta t)-P(t)]$ is the corresponding change in the position of the car. The position P is the **'output variable'** as its value is dependent on the **'input variable'** of time t. Hence we can redefine the numerator as a **'Corresponding change in the output variable'**.

Hence the fundamental definition of a derivative can be stated as the ratio of a **'Corresponding change in the output variable'** and an **'Extremely small change in the input variable'** of a function.

Hence, as stated earlier, Derivative can be defined as a **'rate of change' in an 'output variable' of any function.**

[*Note:- Whenever we say 'rate of change', it is implied that it means 'Instantaneous rate of change', as an input variable Δt is an 'extremely small change in time', $\Delta t \to 0$. However, the word 'Instantaneous' is often omitted, as an input variable can also represent any other variable other than time.*]

We will now use an expression of a general function $f(x)$, instead of expression for a position function $P(t)$ used earlier, to state the fundamental definition of a derivative.

Derivative of function $f(x)$ =

$$= \frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{as } \Delta x \to 0$$

Since, the **derivative** is a **rate of change** in an **output variable** of any function, many other examples like **rate of change** of temperature, or **rate of change** of water level in a tank, or **rate of change** in area of a circle, or **rate of change** in share index, or **rate of change** of population can also be analyzed by applying the principles of **derivative**.

2.5 Comparison- Algebra & Differentiation

The comparison between **Algebra** and **Differentiation** may help us to understand the process of Differentiation even better.

As we know, a **function** is an expression where value of its **'output variable'** is dependent on the varying value of its **'input variable'**. In Algebra, a **function** usually takes a number as an **input**, and give another number as an output.

e.g. : If the **'Doubling function $y = 2x$ '** is given an input of number **3**, then it will give an output of number **6**. But, if the **'Squaring function $y = x^2$ '** is given an input of number **3**, then it will give an output of number **9**.

The **'process of differentiation'**, however takes **'Squaring function'** itself as an input and produces **'another function'** as an output. This means that the **'process of differentiation'** takes all the information of a squaring function – such as 'inputting **2** will give an output of **4**', or 'inputting **3** will give an output of **9**', or 'inputting **4** will give an output of **16**', and so on – and uses this information to produce a new function, called the **'derivative function'** or just the **'derivative'**, which is a **'Doubling function'** in this case, as an output. **e.g. :** If the given function is $f(x) = x^2$, then **derivative** of function $f(x)$ is equal to $2x$.

Note - If the input of the function represents **time**, and the output represents the **position,** then derivative of the function represents **rate of change** in output with respect to **time**. **e.g. :** If $f(x)$ is a function that takes **time** as an input, and gives the **position** of the car at that **time** as an output, then the derivative of $f(x)$ shows how the **position is changing with respect to time.** This means it shows the **'rate of change'** in position or the **velocity** of the car. That is why we said that, when you take a **derivative** of the **position function**, then you get the **velocity function**.

We will see **Graphical, Geometric** and **Algebraic interpretation** of **Derivative** in the subsequent chapters, to bring greater clarity to the **concepts of Differentiation**.

2.6 A Sample of Instantaneous Velocity Calculations

Let us now consider a different instant time of t **= 1.4 min** for our calculations. Let us follow a similar procedure to calculate average velocities of the car for progressively shorter time intervals from instant time of t = **1.4 min** and tabulate these values for the analysis.

Start position time = t (min)	End position time = t_x (min)	Time interval $(t_x - t)$ (min)	Average velocity (km/min)
1.4	2.4	1	3.8
1.4	1.5	0.1	2.9
1.4	1.41	0.01	2.81
1.4	1.401	0.001	2.801
1.399	1.4	0.001	2.799

We can observe that all the values of average velocities are converging to one value of **2.8 km/min**, as we go on reducing the time interval. Hence let us conclude that the **instantaneous velocity** of the car is **2.8 km/min** at the **instant of time t = 1.4 min**.

Let us repeat the calculations to find out average velocities of the car at different instants of time of t **= 2 min** and t **= 3 min** and tabulate those results for the analysis.

Start position time = t (min)	End position time = t_x (min)	Time interval $(t_x - t)$ (min)	Average velocity (km/min)
2	3	1	5
2	2.1	0.1	4.1
2	2.01	0.01	4.01
2	2.001	0.001	4.001
1.999	2	0.001	3.999

Start position time = t (min)	End position time = t_x (min)	Time interval $(t_x - t)$ (min)	Average velocity (km/min)
3	4	1	7
3	3.1	0.1	6.1
3	3.01	0.01	6.01
3	3.001	0.001	6.001
2.999	3	0.001	5.999

We once again observe that the values of average velocities are converging to **4 km/min** and **6 km/min** for progressively shorter time intervals from instants of time of t **= 2** and t **= 3 min** respectively. Hence, let us conclude that **instantaneous velocity** of the car is **4 km/min** at the **instant of time t = 2 min** and **6 km/min** at the **instant of time t = 3 min**.

[*Note:- Speed and velocity are similar terms. The only difference is that, the velocity is indicative of direction also. So, positive velocity would mean that the car is moving in a forward direction, while negative velocity would mean that the car is moving in the reverse direction.*]